On the geometry of decomposition of the cyclic permutation into the product of a given number of permutations

Boris Bychkov (HSE)

Embedded graphs

October 30 2014

BMSH formula

Let σ_0 be a fixed permutation from S_n and take a tuple of c permutations (we don't fix their lengths and quantities of cycles!) from S_n such that $\sigma_1 \cdot \ldots \cdot \sigma_c = \sigma_0$ with some conditions:

- the group generated by this tuple of c permutations acts transitively on the set $\{1, \ldots, n\}$
- $c \cdot n = n + \sum_{i=0}^{c} l(\sigma_i) - 2$ (genus 0 coverings)

BMSH formula

Then there are formula for the number $b_{\sigma_0}(c)$ of such tuples of c permutations by M. Bousquet-Mélou and J. Schaeffer

Theorem (Bousquet-Mélou, Schaeffer, 2000)

Let $\sigma_0 \in S_n$ be a permutation having d_i cycles of length i $(i = 1, 2, 3, \dots)$ and let $l(\sigma_0)$ be the total number of cycles in σ_0, then

$$b_{\sigma_0}(c) = c\frac{(c \cdot n - n - 1)!}{(c \cdot n - n - l(\sigma_0) + 2)!} \prod_{i \geq 1} \left(i \cdot \binom{c \cdot i - 1}{i} \right)^{d_i}$$

Let $n = 3, \sigma_0 = (1, 2, 3), c = 2$, then

$$b_{(1,2,3)}(2) = 2\frac{(2 \cdot 3 - 3 - 1)!}{(2 \cdot 3 - 3 - 1 + 2)!} 3 \binom{2 \cdot 3 - 1}{3} = 5$$

$$\mathrm{id}(123) \quad (123)\mathrm{id} \quad (132)(132) \quad (12)(23) \quad (13)(12) \quad (23)(13)$$

Here are 6 factorizations, but one of them has genus 1!

Main result

Let σ_0 be an n-cycle, then we have a new proof of the Bousquet-Mélou–Schaeffer result (in the case of $\sigma_0 = (1, 2, 3, \ldots, n)$):

Theorem (B, 2014)

The number $b_n(c)$ of c-tuples of permutations such that $\sigma_1 \cdot \ldots \cdot \sigma_c = \sigma_0 = (1, 2, 3, \ldots, n)$ with the two conditions from the first slide is equal to

$$b_n(c) = \frac{1}{n}\binom{cn}{n-1}$$

Continuing our example: $b_3(2) = \frac{1}{3}\binom{6}{2} = 5$

Ramified coverings

The problem of counting decompositions permutation into the product of c permutations is equivalent to the problem of counting ramified coverings of $\mathbb{CP}^1$ by the compact oriented surfaces of genus g with c fixed critical values.

In the case of decomposition n-cycle we automatically have branched cover $f : \mathbb{CP}^1 \mapsto \mathbb{CP}^1$, where f — polynomial of degree n.

The space of polynomials

Let $\mathcal{P}$ be a space of polynomial functions $f : \mathbb{CP}^1 \mapsto \mathbb{CP}^1$.

$$f(x) = x^n + a_2 x^{n-2} + \ldots + a_{n-1}x + a_n$$

Every such a polynomial corresponding to a ramified covering of $\mathbb{CP}^1$ by $\mathbb{CP}^1$ with full ramification over the fixed point ∞ and arbitrary ramifications over other critical values.

Stratification of $\mathcal{P}$

In every critical value we have a partition $\lambda \vdash n$ formed by the lengths of cycles of the corresponding permutation.

General polynomial from $\mathcal{P}$ has $n-1$ simple critical values.
If some critical values coincides polynomial would have less than $n-1$ critical values and some of them would be nonsimple. This polynomials formed *discriminant* in the space $\mathcal{P}$.
Discriminant stratifying by the tuples of partitions corresponding to the critical values.

For example, polynomials with tuple of partitions $\{\lambda_1, \ldots, \lambda_c\}$ (corresponding to the non simple critical values) formed a stratum of the discriminant. Denote it by $\Sigma_{\lambda_1,\ldots,\lambda_c}$.

Lyashko-Looijenga mapping

Lyashko-Looijenga mapping maps polynomial to the set of it's critical values.
The number of the polynomials in the stratum $\Sigma_{\lambda_1,\dots,\lambda_c}$ is equal to the multiplicity of the restriction of the LL mapping on this stratum.

Denote this multiplicity by $\deg_{\lambda_1,\dots,\lambda_c}$, then

Theorem (D. Zvonkine, S. Lando, 1999)

$$\deg_{\lambda_1,\dots,\lambda_c} = n^{c-1}\frac{|\mathrm{Aut}(v(\Pi))|}{|\mathrm{Aut}(\Pi)|}\prod_{i=1}^{c}\frac{|\mathrm{Aut}(v(\lambda_i))|}{|\mathrm{Aut}(\lambda_i)|}$$

Stratum with fixed degeneracies

Let $\lambda = 1^{m_1}2^{m_2}\dots n^{m_n}$, then the *degeneracy* $A(\lambda)$ of this partition (and corresponding critical value) is

$$A(\lambda) = \sum_{i=1}^{n}(i-1)m_i.$$

For example: partitions $1^{n-3}3^1$ and $1^{n-4}2^2$ have the same degeneracy 2.

Fix c and let $k_1, \dots, k_c$ — nondecreasing sequence of nonnegative integers: $k_1 \geq k_2 \geq \dots k_c \geq 0$, $k_1 + k_2 + \dots + k_c = n-1$.

Denote by $\Sigma_{k_1,\dots,k_c}$ the discriminant stratum of the $\mathcal{P}$ formed by polynomials with the set of degeneracies of the finite critical values coincides with the nonzero elements in the tuple $\{k_1, \dots, k_c\}$.

Proof of the main result

Lemma

The multiplicity of the restriction LL on the stratum with one nonsimple critical value with the degeneracy k given by the formula

$$\deg_k = n^{n-k-2}\binom{n}{k}$$

Lemma

The multiplicity of the restriction LL on the stratum $\Sigma_{k_1,\dots,k_c}$ given by the formula

$$\deg_{k_1,\dots,k_c} = \frac{1}{n}\binom{n}{k_1}\binom{n}{k_2}\cdots\binom{n}{k_{c-1}}\binom{n}{k_c}$$

Proof of the main result

We obtain the main result from the second lemma by summarizing over the all not ordered tuples $\{k_1, \ldots, k_c\}$ such that $k_1 + \ldots + k_c = n - 1$.

Thank you!

Bon appetite!